

Monday, Sep 29, Lecture 13: Complex Numbers

Last time ... Optimization and Lagrange Multipliers.

Complex Numbers

Introduce a "number" i such that

$$i^2 = -1,$$

Def: complex numbers $\mathbb{C}$

The system of complex numbers $\mathbb{C}$ is

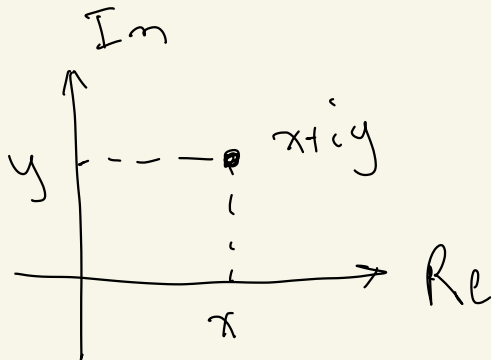
• The ^(real) vector space $\mathbb{R}^2 = \mathbb{C}$, so

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \quad \leftarrow \text{Just vector addition}$$

$$a(x + iy) = ax + iay, \quad \forall a \in \mathbb{R}$$

• with multiplication

$$\begin{aligned} (x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 + iy_1x_2 + iy_2x_1 + i^2y_1y_2 \\ &= x_1x_2 - y_1y_2 + i(y_1x_2 + y_2x_1) \end{aligned}$$



Let $z = x + iy$.

"Real part" $\text{Re}(z) = x$ $\leftarrow$ so, $\text{Re}: \mathbb{C} \rightarrow \mathbb{R}$

"Imaginary part" $\text{Im}(z) = y$ $\leftarrow$ $\text{Im}: \mathbb{C} \rightarrow \mathbb{R}$

Def: Polar form (part 1)

Let $z = a + bi$. The norm $|z|$ is a real number

$$|z| = \sqrt{a^2 + b^2}$$

aka modulus or absolute value or magnitude (yes, it's a lot.)

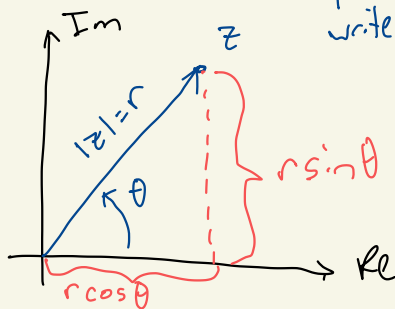
Same as norm on $\mathbb{R}^2$,

$$|z|^2 = a^2 + b^2$$

aka angle, phase

and the argument is the angle $\theta \in [0, 2\pi)$ it makes with the real axis:

write: $\arg(z) = \theta$.



$$\begin{aligned} z &= r \cos \theta + i r \sin \theta \\ &= r (\cos \theta + i \sin \theta) \\ &= re^{i\theta} \text{ (later)} \end{aligned}$$

* We have demanded $\theta \in [0, 2\pi)$ to remove ambiguity. But we could just as easily have asked $\theta \in [-\pi, \pi)$ or $\theta \in [8\pi, 10\pi)$. We call the choice $[a, a + 2\pi)$ a branch of the argument.

Ex: find the polar form of

$$z = 1 + i\sqrt{3}$$

Soln:

$$|z| = \sqrt{1 + 3} = \sqrt{4} = 2$$

$$\arg(z) = \arctan\left(\frac{\sqrt{3}}{1}\right) = \frac{\pi}{3}$$

$\sqrt{3}$



To A:

$$\tan \theta = \frac{\sqrt{3}}{1}$$

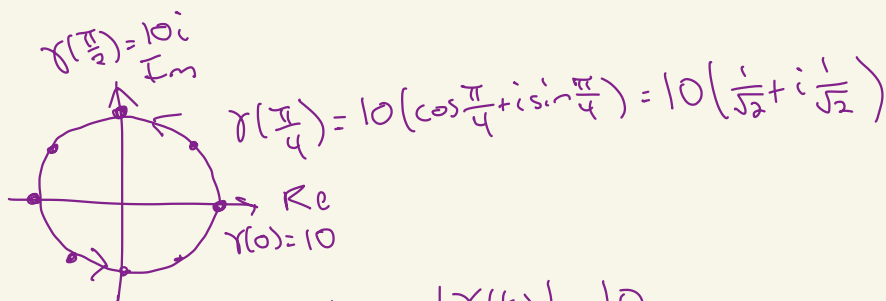
$$\Rightarrow z = 2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$$

$\uparrow$ $\uparrow$
 $r (\cos \theta + i \sin \theta)$

Ex: what path is traced out by $\gamma: [0, 2\pi) \rightarrow \mathbb{C}$

$$\gamma(t) = 10(\cos t + i \sin t) \quad ?$$
$$= 10e^{it}$$

Soln:



and this makes sense since $|\gamma(t)| = 10$
and $\arg(\gamma(t)) = t$

- Rectangular form $z = x + iy$ is nice for "vector space stuff", like addition and subtraction.
- Polar form is nice for multiplication:

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2}$$
$$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

⇒ Proposition:

- $|z_1 z_2| = |z_1| |z_2|$
- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) \pmod{2\pi}$

Geometrically,

Scaling:

If $z_1 = |z_1|$,

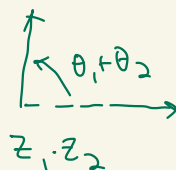
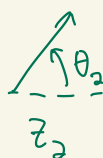
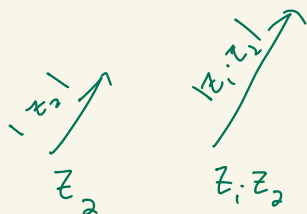
Note: $\arg(z_1) = 0$

Rotation:

If $z_1 = \cos \theta_1 + i \sin \theta_1$,

Note:

$|z_1| = 1$



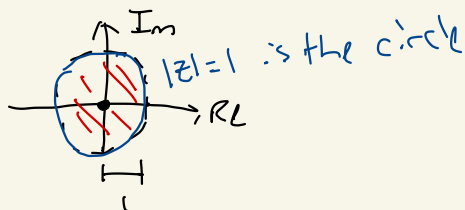
Ex:

Find all solutions to $|z| \leq 1$ and $|z| < 1$

Soln:

Well, $z = a + ib$ has $|z|^2 = a^2 + b^2$

$\Rightarrow$ $|z| < 1$ means distance from 0 is less than 1



So: $\{z : |z| \leq 1\}$ = closed disc of radius 1, centered at $z=0$

and

$\{z : |z| < 1\}$ = open disc of radius 1, centered at $z=0$



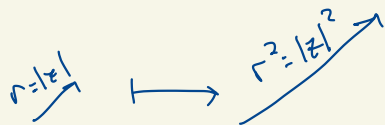
Ex:

Define $f: \mathbb{C} \rightarrow \mathbb{C}$ by $f(z) = z^2$. How to think about it?

Well... $(a+bi)^2 = a^2 - b^2 + i2ab$ is just confusing.

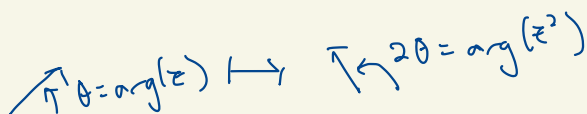
But

$$|z^2| = |z|^2 \leftarrow \text{squares magnitude}$$

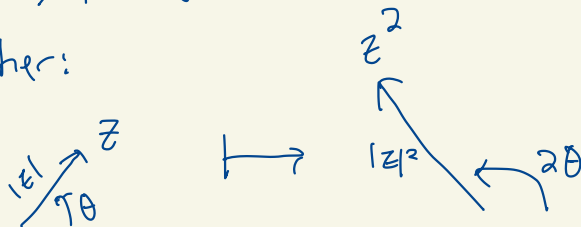


while

$$\begin{aligned} \arg(z^2) &= \arg(z \cdot z) = \arg(z) + \arg(z) \pmod{2\pi} \\ &= 2\arg(z) \pmod{2\pi} \end{aligned}$$



All together:



Ex: Find all solns to $z^2 = 2i$.

Soln:

$$2i = (a+bi)^2 = a^2 + 2abi - b^2$$

Just like vector equations, this is secretly 2 eqns (real & imaginary)

$$0 = a^2 - b^2 \Rightarrow a^2 = b^2$$

$$2i = 2abi \Rightarrow ab = 1$$

2 solns:
 $a=b=1 \Rightarrow 1+i$
 $a=b=-1 \Rightarrow -1-i$

